A C^∞-REGULARITY THEOREM FOR NONDEGENERATE CR MAPPINGS

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ABSTRACT. We prove the following regularity result: If $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ are smooth generic submanifolds and M is minimal, then every C^k-CR-map from M into M' which is k-nondegenerate is smooth. As an application, every CR diffeomorphism of k-nondegenerate minimal submanifolds in $\mathbb{C}^N$ of class C^k is smooth.

1. Introduction and statement of results

We first briefly describe the setting for the results which we want to discuss. Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be generic, real submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. We shall denote by d the real codimension of M and by d' the real codimension of M', and write $n = N-d$, $n' = N'-d'$. Recall that M is generic if there is a smooth defining function $\rho = (\rho_1, \dots, \rho_d)$ for M such that the vectors $\rho_{1,Z}(p), \dots, \rho_{d,Z}(p)$ are linearly independent for $p \in M$. Here for any smooth function ϕ we let $\phi_Z = (\frac{\partial \phi}{\partial Z_1}, \dots, \frac{\partial \phi}{\partial Z_N})$ be its complex gradient.

We also fix points $p_0 \in M$ and $p_0' \in M'$ (which we will assume to be equal to 0 for most of this paper). A C^k-mapping H from M into M' is said to be CR if its differential dH satisfies $dH(T_p^c M) \subset T^c_{H(p)}M'$ for $p \in M$, where $T_p^c M$ denotes the complex tangent space to M at p, that is, the largest subspace of the real tangent space T_pM invariant under the complex structure operator J in $\mathbb{C}^N$. Equivalently, if $H = (H_1, \dots, H_{N'})$ for any system of holomorphic coordinates in $\mathbb{C}^{N'}$, each H_j is a CR-function on M. (For further reference on these definitions, the reader is referred to the book of Baouendi, Ebenfelt and Rothschild [1]).

The following definition is from [9]. We shall give it in a slightly modified form.

Definition 1. *Let M, M' be as above. Let $\rho' = (\rho_1', \dots, \rho_{d'}')$ be a defining function for M' near $H(p_0)$, and choose a basis $L_1, \dots, L_n$ of CR-vector fields tangent to M near p_0. We shall write $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$ for any multiindex α. Let $H : M \to M'$ be a CR-map of class C^m. For $0 \le k \le m$, define the increasing sequence of subspaces $E_k(p_0) \subset \mathbb{C}^{N'}$ by*

$$E_k(p_0) = \operatorname{span}_{\mathbb{C}}\{L^\alpha \rho'_{l,Z'}(H(Z), \overline{H(Z)})|_{Z=p_0} \colon 0 \le |\alpha| \le k, 1 \le l \le d'\}. \tag{1}$$

We say that H is k_0-nondegenerate at p_0 (with $0 \le k_0 \le m$) if $E_{k_0-1}(p_0) \neq E_{k_0}(p_0) = \mathbb{C}^{N'}$.

The invariance of this definition under the choices of the defining function, the basis of CR vector fields and the choices of holomorphic coordinates in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ is easy to show; the reader can find proofs for this in [9] or [8].

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Recall that if $\Gamma \subset \mathbb{R}^d$ is an open convex cone, $p_0 \in M$, and $U \subset \mathbb{C}^N$ is an open neighbourhood of p_0, then a wedge W with edge M centered at p_0 is defined to be a set of the form $W = \{Z \in U : \rho(Z, \bar{Z}) \in \Gamma\}$, where ρ is a local defining function for M. We can now state our main theorem.

Theorem 2. *Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be smooth generic submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, $p_0 \in M$ and $p_0' \in M'$, $H : M \to M'$ a C^{k_0}-CR-map which is k_0-nondegenerate at p_0 and extends continuously to a holomorphic map in a wedge W with edge M. Then H is smooth in some neighbourhood of p_0.*

This theorem is the smooth version of the main result in [9]. Let us recall that M is said to be minimal at p_0 if there does not exist any CR-submanifold through p_0 strictly contained in M with the same CR dimension as M. By a theorem of Tumanov, if M is minimal, every continuous CR-function f on M near p extends continuously to a holomorphic function into a wedge W with edge M. Hence we have the following corollary.

Corollary 3. *Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be smooth generic submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, $p_0 \in M$ and $p_0' \in M'$, M minimal at p_0, $H : M \to M'$ a C^{k_0} map which is k_0-nondegenerate at p_0. Then H is smooth in some neighbourhood of p_0.*

Note that by a regularity theorem of Rosay ([13], see also [1]), if the boundary value of a holomorphic function in a wedge W with edge M is C^k on M, then the extension is also of class C^k up to the edge. Hence, for the proof of Theorem 2 we will assume that H extends in a C^{k_0}-fashion to a wedge W centered at p_0.

We would like to mention one particular instance of this theorem. If M is a manifold whose identity map is k_0-nondegenerate in the sense of Definition 1, then we say that M is *k_0-nondegenerate*. This notion has been introduced for hypersurfaces by Baouendi, Huang and Rothschild in [2]; for a thorough introduction to this nondegeneracy condition for submanifolds and its connection with holomorphic nondegeneracy in the sense of Stanton ([15]), see [1], or the paper of Ebenfelt [5]. In particular, every CR-diffeomorphism of class C^{k_0} of a k_0-nondegenerate submanifold is k_0-nondegenerate in the sense of Definition 1. Theorem 2 implies the following regularity result for k_0-nondegenerate smooth submanifolds.

Corollary 4. *Assume that $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ are k_0-nondegenerate smooth submanifolds of real codimension d, M minimal at p_0, and $H : M \to M'$ is a CR-diffeomorphism of class C^{k_0}. Then H is smooth.*

If $d = 1$, we can drop the assumption of minimality, since in the hypersurface case, k_0-nondegeneracy implies minimality. In the case where $N = N' = 2$ and $d = 1$, Corollary 4 is basically contained in the thesis of Roberts [12]. The Levi-nondegenerate hypersurface case is well understood; the connection with the results proved in this paper is that Levi-nondegeneracy of hypersurfaces is equivalent to 1-nondegeneracy. In fact, for Levi-nondegenerate hypersurfaces, Corollary 4 is due to Nirenberg, Webster and Yang [10], and of course we should not forget to mention Fefferman's mapping theorem [6] (however, we shall not deal with the C^1-extension here). A proof for strictly pseudoconvex hypersurfaces of finite smoothness was given by Pinchuk and Khasanov [11]. More recently, Tumanov [16] has proved the corresponding theorem for Levi-nondegenerate targets of higher codimension. For results for pseudoconvex targets, we want to refer the reader to the historical

discussion in the paper by Coupet and Sukhov [4] and the newer results for convex hypersurfaces by Coupet, Gaussier and Sukhov [3].

The paper is organized as follows. In section 2, 3, and 4 we present the technical foundations for the proof. Although these results are well known, they are not easy to find in the literature; so, in order to make this paper as self contained as possible, we have decided to include the proofs. Theorem 2 is then proved in section 5.

2. Boundary values of functions of slow growth

In this section, we will develop an integral representation for a $\bar\partial$-bounded function of slow growth (in a wedge with straight edge). Let us first fix notation. Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{R}^d$ be open subsets, and let $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{R}^d$ with $0 < \delta_j$ for $0 \le j \le d$. We set $\Omega_+ = \{(z,s,t) \in U \times V \times \mathbb{R}^d \colon 0 < t < \delta\}$, $\Omega_- = \{(z,s,t) \in U \times V \times \mathbb{R}^d \colon 0 > t > -\delta\}$ and $\Omega_0 = U \times V \times \{0\}$, and we will write $z = (x,y)$ for the underlying real variables. Throughout the paper, dm will denote Lebesgue measure. Let $\mathfrak{B}(\Omega_+)$ be the space of all functions $h \in C^1(\Omega_+)$ that extend smoothly to the set $E = \{(z,s,t) \in \bar\Omega_+ \colon t \ne 0\}$ which have the following property: For each compact set $K \subset U \times V$, there exist positive constants C_1, μ and C_2 (depending on K and h) such that

$$\sup_{(z,s)\in K, 0<t<\delta} |t|^\mu |h(x,y,s,t)| \le C_1 \tag{2}$$

and

$$\sup_{(z,s)\in K, 0<t<\delta} |\bar\partial_j h(x,y,s,t)| \le C_2, \quad 1 \le j \le d. \tag{3}$$

Here we write $\bar\partial_j = \frac{1}{2}\left(\frac{\partial}{\partial s_j} + i\frac{\partial}{\partial t_j}\right)$. We have the following (probably well known) result, which we state for $\mathfrak{B}(\Omega_+)$; however, we define $\mathfrak{B}(\Omega_-)$ in a similar manner, and all the results stated in this section hold equally well for $\mathfrak{B}(\Omega_-)$.

Theorem 5. *Let $h \in \mathfrak{B}(\Omega_+)$. Then the limit*

$$\langle b_+ h, \phi\rangle = \lim_{\epsilon=(\epsilon_1,\dots,\epsilon_d)\to 0} \int_{U\times V} h(x,y,s,\epsilon)\phi(x,y,s)\, dm \tag{4}$$

exists for each $\phi \in C_c^\infty(\Omega_0)$ and defines a distribution b_+h called the boundary value of h. Furthermore, for each compact set K there exists an integer v_0 such that for $v \ge v_0$, for each $j = 1, \dots, d$, $0 \le \delta' \le \delta_j$ we have the following integral representation for $\phi \in C_c^\infty(U \times V)$ with $\operatorname{supp}\phi \subset K$*:*

$$\begin{aligned} \langle b_+ h, \phi\rangle &= \int_{U\times V} h(x,y,s,0,\dots,\delta',\dots,0) S_v\phi(x,y,s,0,\dots,\delta',\dots,0)\, dm \\ &+ 2i \int_{U\times V}\int_0^{\delta'} \bar\partial_1 h(x,y,s,0,\dots,t_j,\dots,0) S_v\phi(x,y,s,0,\dots,t_j,\dots,0)\, dt_j dm \\ &+ 2i \int_{U\times V}\int_0^{\delta'} h(x,y,s,0,\dots,t_j,\dots,0) D_{s_j}^{v+1}\phi(x,y,s) t_j^v\, dt_j dm. \end{aligned} \tag{5}$$

where

$$S_v\phi(x,y,s,t) = \sum_{|\alpha|\le v} \frac{1}{\alpha!} D_s^\alpha \phi(x,y,s) t^\alpha. \tag{6}$$

Proof. Let $S_v\phi$ be defined by (6). We are going to prove the formula under the assumption that $j = 1$. Fix (x, y), $s_2, \dots s_d$ and $0 < \delta' < \delta_1$, and assume $0 < \epsilon_1 < \delta_1 - \delta'$. First we are going to assume that $K = \operatorname{supp}\phi$ is contained in a product of the form $U_1 \times [a, b] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ contained in a relatively compact open subset $W \subset U \times V$. In this case, define

$$u(s_1, t_1) = h(x, y, s_1, s_2, \dots, s_d, \epsilon_1 + t_1, \epsilon_2, \dots, \epsilon_d) S_v\phi(z, s, t_1, 0, \dots, 0).$$

Clearly, u is C^1 on the square $\omega = [a, b] \times [0, \delta']$ and $u(s_1, t_1) = 0$ if $s_1 \geq b$ or $s_1 \leq a$. By Stokes formula,

$$\int_{\partial\omega} u(s_1, t_1)\, dw = 2i \int_\omega \bar\partial u(s_1, t_1)\, dm,$$

where we have set $w = s_1 + it_1$ and $\bar\partial = \bar\partial_1$. This formula translates into

$$\begin{aligned}(7)\quad &\int_a^b h(x, y, s, \epsilon)\phi(x, y, s)\, ds_1 = \\ &\qquad\int_a^b h(x, y, s, \epsilon_1 + \delta', \epsilon_2, \dots, \epsilon_d) S_v\phi(x, y, s, \delta', 0, \dots, 0)\, ds_1 \\ &\quad + 2i \int_0^{\delta'} \int_a^b \bar\partial_1 h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \dots, \epsilon_d) S_v\phi(x, y, s, t_1, 0, \dots, 0)\, ds_1 dt_1 \\ &\qquad\quad + 2i \int_0^{\delta'} \int_a^b \bar\partial_1 h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \dots, \epsilon_d) D_{s_1}^{v+1}\phi(x, y, s) t_1^v\, ds_1 dt_1.\end{aligned}$$

We integrate this formula with respect to $(x, y, s_2, \dots, s_d)$ to obtain

$$\begin{aligned}(8)\quad &\int_W h(x, y, s, \epsilon)\phi(x, y, s)\, dm = \\ &\qquad\int_W h(x, y, s, \epsilon_1 + \delta', \epsilon_2, \dots, \epsilon_d) S_v\phi(x, y, s, \delta', 0, \dots, 0)\, dm \\ &\quad + 2i \int_W \int_0^{\delta'} \bar\partial_1 h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \dots, \epsilon_d) S_v\phi(x, y, s, t_1, 0, \dots, 0)\, dt_1 dm \\ &\qquad\quad + 2i \int_W \int_0^{\delta'} h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \dots, \epsilon_d) D_{s_1}^{v+1}\phi(x, y, s) t_1^v\, dt_1 dm.\end{aligned}$$

For each of these integrals, we can use the bounded convergence theorem to take the limit as $\epsilon \to 0$, provided that we choose $v \geq \mu_K$, where μ_K denotes the least integer μ for which (2) holds on K and to obtain an estimate of the form $|\langle b_+ h, \phi\rangle| \leq C\|\phi\|_{v+1}$ (where $\|\phi\|_k = \max_{x \in U\times V, |\alpha| \leq k} |\phi^\alpha(x)|$).

Now we pass to the case of general K by covering with finitely many sets of the form considered above and using a partition of unity. The details are easy and left to the reader. $\square$

Consider now the class $\mathfrak{A}(\Omega_+)$ of functions h which are smooth on E with the property that for all α, β we have that $D^\alpha_{x,y} D^\beta_s h \in \mathfrak{B}(\Omega_+)$. If $h \in \mathfrak{A}(\Omega_+)$, for $K \subset U \times V$ we let $\mu_l(h, K)$ the smallest integer μ such that

$$\sup_{(z,s)\in K, 0<t<\delta} |t|^\mu |D^\alpha_{x,y} D^\beta_s h(x, y, s, t)| \leq C_1, \quad |\alpha| + |\beta| \leq l \tag{9}$$

for some constant C_1. Let us also introduce the space $\mathfrak{A}_\infty(\Omega_+)$ of functions in $\mathfrak{A}(\Omega_+)$ with the additional property that for any compact set $K \subset U \times V$, for any

multiindeces α and β, and for any nonnegative integer k there exists a constant C such that

$$\sup_{(z,s)\in K, 0<t<\delta} |D^\alpha_{x,y} D^\beta_s \bar\partial_j h(x,y,s,t)| \le C|t|^k, \quad 1 \le j \le d. \tag{10}$$

Of course, we define the spaces $\mathfrak{A}(\Omega_-)$ and $\mathfrak{A}_\infty(\Omega_-)$ analogously, and the results stated below for $\mathfrak{A}(\Omega_+)$ and $\mathfrak{A}_\infty(\Omega_+)$ also hold for $\mathfrak{A}(\Omega_-)$ and $\mathfrak{A}_\infty(\Omega_-)$. This can be seen most easily by noting the following useful fact: If $h(x,y,s,t) \in \mathfrak{A}(\Omega_+)$ (or $\mathfrak{A}_\infty(\Omega_+)$, respectively), $\overline{h(x,y,s,-t)} \in \mathfrak{A}(\Omega_-)$ (or $\mathfrak{A}_\infty(\Omega_-)$, respectively).

We will also need the space of functions which are *almost holomorphic* on $U \times V$. This is the space

$$\mathfrak{AH}(U\times V) = \{a \in C^\infty(U\times V\times\mathbb{R}^d)\colon D^\alpha_{x,y} D^\beta_s D^\gamma_t \bar\partial_j a(x,y,s,0) = 0, 1\le j\le d\}. \tag{11}$$

Lemma 6. *Let $h \in \mathfrak{A}(\Omega_+)$, $a \in \mathfrak{AH}(U\times V)$, and set $a_0(x,y,s) = a(x,y,s,0)$. Then $ah \in \mathfrak{A}(\Omega_+)$, and $b_+ah = a_0 b_+ h$ in the sense of distributions. Furthermore, if $h \in \mathfrak{A}_\infty(\Omega_+)$, so is ah.*

Proof. By the Leibniz rule, $D^\alpha_{x,y} D^\beta_s ah$ is a sum of products of derivatives of a and h. It is clear that such a sum fulfills (2). To see that it also fulfills (3), note that by (11) every derivative of $\bar\partial_j a$ vanishes to infinite order on $t=0$.

To see that $b_+ah = a_0b_+h$ we use Taylor development to write $a(x,y,s,t) = \sum_{|\beta|\le k} \frac{1}{\beta!} D^\beta_s a(x,y,s,0)(it)^\beta + O(|t|^{k+1})$ (uniformly on compact subsets of $U\times V$). Now choose $k \ge \mu_0(h,K)$ and substitute into (4) for ϕ with $\operatorname{supp}\phi \subset K$. The claim follows now by taking the limit and using Theorem 5. ☐

Basically the same proof shows the following Lemma.

Lemma 7. *Assume that X is a vector field on $U\times V\times\mathbb{R}^d$ which is tangent to all subspaces of the form $t=c$, where $c\in\mathbb{R}^d$ is a constant vector, and such that all the coefficients of X are in $\mathfrak{AH}(U\times V)$. Set $X_0 = X|_{t=0}$. If $h \in \mathfrak{A}(\Omega_+)$, then $Xh \in \mathfrak{A}(\Omega_+)$, and $b_+Xh = X_0b_+h$ in the sense of distributions. Furthermore, if $h\in\mathfrak{A}(\Omega_+)$, so is Xh.*

3. An almost holomorphic edge-of-the-wedge theorem

The main result of this section is the following theorem. Our presentation follows closely [12], but we also want to refer the reader to [14]. We keep the notation from the proceeding section and since we shall use the Fourier transform we also introduce the following new variables: $\xi\in\mathbb{R}^n$, $\tau\in\mathbb{R}^n$, $\sigma\in\mathbb{R}^d$. For a distribution ϕ on $U\times V$ we will write $\hat\phi(\xi,\tau,\sigma) = \langle \phi, \exp(-i(x\xi+y\tau+s\sigma))\rangle$ for its Fourier transform.

Theorem 8. *Assume that $h_+ \in \mathfrak{A}(\Omega_+)$, $h_- \in \mathfrak{A}(\Omega_-)$, and that $b_+h_+ = b_-h_- = h$. Then h is smooth.*

The proof follows from the next Lemma.

Lemma 9. *Let $h\in\mathfrak{A}(\Omega_+)$, and $\phi\in C^\infty_c(U\times V)$. Then for every $k\in\mathbb{N}$ there exists a constant C_k such that if $\zeta = (\xi,\tau,\sigma)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d$ with $\sigma_j \le 0$ for some j, $1\le j\le d$, then*

$$|\widehat{\phi b_+ h}(\zeta)| \le \frac{C_k}{(1+|\zeta|^2)^k}. \tag{12}$$

Here, C_k depends on k, ϕ, and h. The same result holds with $\mathfrak{A}(\Omega_+)$ replaced by $\mathfrak{A}(\Omega_-)$ if $\sigma_j \geq 0$ for some j.

Proof. For the moment, fix ζ; for simplicity, assume that $j = 1$, so that $\sigma_1 \leq 0$. We shall write $a(x,y,s,t) = \exp(-i(x\xi + y\tau + s\sigma) + t\sigma)$. Then $a \in \mathfrak{A}\mathfrak{H}(U \times V)$—in fact, $\partial_j a = 0$, $1 \leq j \leq d$. We let Δ be the real Laplacian in the $2n + d$ variables (x, y, s), that is,

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} + \sum_{j=1}^{d} \frac{\partial^2}{\partial s_j^2}. \tag{13}$$

We then have that $(1+\Delta)^k a(x,y,s,t) = (1+|\zeta|^2)^k a(x,y,s,t)$. Recall that we write $a_0(x,y,s) = a(x,y,s,0)$. By Lemma 6, we see that $\widehat{\phi b_+ h}(\zeta) = \langle \phi b_+ h, a_0\rangle = \langle b_+ h, \phi a_0\rangle = \langle a_0 b_+ h, \phi\rangle = \langle b_+ a h, \phi\rangle$. We apply the integral formula (5) from Theorem 5 for $j = 1$, and some δ', which implies that

$$\begin{aligned}
(14)\quad \langle b_+ a h, \phi\rangle = &\\
&\int_{U\times V} h(x,y,s,\delta',0) e^{-i(x\xi+y\tau+s\sigma)} e^{\delta'\sigma_1} S_v\phi(x,y,s,\delta',0)\, dm\\
&+ 2i \int_{U\times V}\int_0^{\delta'} (\bar\partial_1 h(x,y,s,t_1,0)) e^{-i(x\xi+y\tau+s\sigma)} e^{t_1\sigma_1} S_v\phi(x,y,s,t_1,0)\, dt_1 dm\\
&+ 2i \int_{U\times V}\int_0^{\delta'} h(x,y,s,t_1,0) e^{-i(x\xi+y\tau+s\sigma)} e^{t_1\sigma_1} D_{s_1}^{v+1}\phi(x,y,s) t_1^v\, dt_1 dm\\
&= I_1 + I_2 + I_3.
\end{aligned}$$

We now replace $e^{-i(x\xi+y\tau+s\sigma)}$ by $\frac{1}{(1+|\zeta|^2)^k}(1+\Delta)^k e^{-i(x\xi+y\tau+s\sigma)}$ in all three integrals above. Then we integrate by parts and estimate, where we choose $v \geq \mu_{2k}(h, K)$ (see (9) for the definition of this number) with $K = \operatorname{supp}\phi$. Since all the estimates are easy, we do not write them out; the reader can easily check them. □

Proof of Theorem 8. Let $p \in U \times V$. Choose a function $\phi \in C_c^\infty(U \times V)$ which is equal to 1 in some open neighbourhood of p. By Lemma 9, since $h_+ \in \mathfrak{A}(\Omega_+)$ and $h_- \in \mathfrak{A}(\Omega_-)$, we have that

$$|\widehat{\phi h}(\zeta)| \leq \frac{C_k}{(1+|\zeta|^2)^k}. \tag{15}$$

for all $\zeta \in \mathbb{R}^{2n+d}$. Hence, ϕh is smooth (see for Example [7]), and so h is smooth in some neighbourhood of p, since $\phi \equiv 1$ there. Since p was arbitrary, the claim follows. □

4. A version of the Implicit Function Theorem

We will need the following, "almost holomorphic", implicit function theorem.

Theorem 10. *Let $U \subset \mathbb{C}^N$ be open, $0 \in U$, $A \in \mathbb{C}^p$, $F : U \times \mathbb{C}^p \to \mathbb{C}^N$ be smooth in the first N variables and polynomial in the last p variables, and assume that $F(0, A) = 0$ and $F_Z(0, A)$ is invertible. Then there exists a neighbourhood $U' \times V'$ of $(0, A)$ and a smooth function $\phi : U' \times V' \to \mathbb{C}^N$ with $\phi(0, A) = 0$, such that if*

$F(Z,\bar Z,W)=0$ for some $(Z,W)\in U'\times V'$, then $Z=\phi(Z,\bar Z,W)$. Furthermore, for every multiindex α, and each j, $1\le j\le N$,

$$D^\alpha \frac{\partial \phi_j}{\partial Z_k}(Z,\bar Z,W)=0,\quad 1\le k\le N, \tag{16}$$

if $Z=\phi(Z,\bar Z,W)$, and ϕ is holomorphic in W. Here, D^α denotes the derivative in all the real variables.

Proof. Let us write $F(Z,\bar Z,W)=F(x,y,W)$ where $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ are the underlying real coordinates in $\mathbb{C}^N$, as usual identified by $Z_j=x_j+iy_j$. Let us also choose a neighbourhood $U_0\subset\mathbb{R}^N$ of 0 with the property that $U_0\times U_0\subset U$. We extend F in the first $2N$ variables almost holomorphically; that is, we have a function $\tilde F: U_0\times\mathbb{R}^N\times U_0\times\mathbb{R}^N\times\mathbb{C}^p\to\mathbb{C}^N$ with the property that

$$\tilde F(x,x',y,y',W)|_{x'=y'=0}=F(x,y,W) \tag{17}$$

and, if we introduce complex coordinates $\xi_k=x_k+ix'_k$, $\eta_k=y_k+iy'_k$, $1\le k\le N$, then

$$D^\alpha \frac{\partial \tilde F_j}{\partial \bar\xi_k}\bigg|_{x'=y'=0}=D^\alpha \frac{\partial \tilde F_j}{\partial \bar\eta_k}\bigg|_{x'=y'=0}=0,\quad 1\le j,k\le N. \tag{18}$$

Also, $\tilde F$ is still polynomial in W. We introduce new coordinates $\chi=(\chi_1,\dots,\chi_N)\in\mathbb{C}^N$ by

$$\xi_k=\frac{z_k+\chi_k}{2},\quad \eta_k=\frac{z_k-\chi_k}{2i},\quad 1\le k\le N,$$

and write $G(Z,\bar Z,\chi,\bar\chi,W)=F(\xi,\bar\xi,\eta,\bar\eta,W)$. G is smooth in the first $2N$ complex variables in some neighbourhood of the origin, and polynomial in W. We will now compute the real Jacobian of G with respect to Z at (O,A). At $(0,A)$, $\frac{\partial G}{\partial Z}(0,A)=\frac{\partial F}{\partial Z}(0,A)$ and $\frac{\partial G}{\partial \bar Z}(0,A)=0$, so that we have

$$\det\begin{pmatrix}\frac{\partial G}{\partial Z} & \frac{\partial G}{\partial \bar Z}\\ \frac{\partial \bar G}{\partial Z} & \frac{\partial \bar G}{\partial \bar Z}\end{pmatrix}(0,A)=\left|\det\frac{\partial F}{\partial Z}(0,A)\right|^2\ne 0$$

by assumption. Hence, by the implicit function theorem, there exists a smooth function ψ defined in some neighbourhood of $(0,A)$, valued in $\mathbb{C}^N$, such that $Z=\psi(\chi,\bar\chi,W)$ solves the equation $G(Z,\bar Z,\chi,\bar\chi,W)=0$ uniquely. Here we have already taken into account that ψ depends holomorphically on W, a fact that the reader will easily check. Since $G(Z,\bar Z,\bar Z,Z,W)=F(Z,\bar Z,W)$, this implies that if $F(Z,\bar Z,W)=0$, then $Z=\psi(\bar Z,Z,W)$.

We let $\phi(Z,\bar Z,W)=\psi(\bar Z,Z,W)$ and claim that ϕ satisfies (16). In fact, computation shows that $\phi_Z(Z,\bar Z,W)=\psi_{\bar\chi}=-(G_Z-G_{\bar Z}\bar G_{\bar Z}^{-1}\bar G_Z)^{-1}(G_{\bar\chi}+G_{\bar Z}\bar G_{\bar Z}^{-1}\bar G_{\bar\chi})$, where the right hand side is evaluated at $(\psi(\bar Z,Z,W),\bar\psi(\bar Z,Z,W),\bar Z,Z,W)$. This formula shows that each ϕ_{j,Z_k} is a sum of products each of which contains a factor which is a derivative of G with respect to $\bar Z$ or $\bar\chi$.

By the definition of G, we have that

$$\frac{\partial G}{\partial \bar Z}=\frac12\frac{\partial \tilde F}{\partial \bar\xi}+\frac{1}{2i}\frac{\partial \tilde F}{\partial \bar\eta},\quad \frac{\partial G}{\partial \bar\chi}=\frac12\frac{\partial \tilde F}{\partial \bar\xi}-\frac{1}{2i}\frac{\partial \tilde F}{\partial \bar\eta}.$$

By (18) every derivative of those vanishes if $x'=y'=0$, which is in turn the case if $\operatorname{Im}\frac{\phi(\bar Z,Z)+\bar Z}{2}=0$ and $\operatorname{Im}\frac{\phi(\bar Z,Z)-Z}{2i}=0$. But this is clearly fulfilled if $Z=\phi(\bar Z,Z)$.

The proof is now finished by applying the Leibniz rule, the chain rule and the observations made above. □

Note that it is clear from the usual implicit function theorem that we can solve for N of the real variables (x,y). What this theorem asserts is that we can do so in a special manner.

5. Proof of Theorem 2

Let us start by choosing coordinates. There is a neighbourhood U of $p_0 = 0$ in $\mathbb{C}^N$ and a smooth function $\phi : \mathbb{C}^n \times \mathbb{R}^d \to \mathbb{R}^d$ defined in a neighbourhood V of 0 such that $M \cap U = \{(z, s + i\phi(z,\bar z, s)) \colon (z,s) \in V\}$ with the property that $\nabla\phi(0) = 0$. Since the conclusion of the theorem is local, we shall replace M by $M \cap U$, and use this representation. For suitably chosen open sets $U \subset \mathbb{C}^n$ and $V \subset \mathbb{R}^d$, consider the diffeomorphism $\Psi : U \times V \to M$, $\Psi(z,\bar z,s) = (z, s + i\phi(z,\bar z,s))$. We extend this diffeomorphism almost holomorphically to a map, again denoted by Ψ, from $U \times V \times \mathbb{R}^d$ to $\mathbb{C}^N$. Ψ is a diffeomorphism in an open neighbourhood of $U \times V \times \{0\}$, and it has the property that for every component Ψ_l of Ψ,

$$D^\alpha_{x,y,s,t} \bar\partial_j \Psi_l(z,s,0) = 0, \quad (z,s) \in U \times V, \tag{19}$$

where the derivative is in all the real variables. Equivalently,

$$D^\alpha_{x,y} D^\beta_s \bar\partial_j \Psi_l(z,s,0) = O(|t|^\infty), \quad (z,s) \in U \times V, \tag{20}$$

uniformly on compact subsets of $U \times V$. That is, for each α, β, $K \subset U \times V$ compact and every $l \in \mathbb{N}$ there exists a constant $C_l = C_l(\alpha,\beta,K)$ such that

$$|D^\alpha_{x,y} D^\beta_s \bar\partial_j \Psi_l(z,s,t)| \le C_l |t|^l, \quad (z,s) \in K. \tag{21}$$

We assume that each component H_j of H extends continuously (and, consequently by a theorem of Rosay [13] already alluded to above, in a C^k-fashion) to a holomorphic function into a wedge with edge M. Let us recall that this means that with an open convex cone Γ in $\mathbb{R}^d$ each H_j extends continuously to the set $W_\Gamma = \{Z \in U_0 \colon \rho(Z,\bar Z) \in \Gamma\}$, where U_0 is an open neighbourhood of 0 in $\mathbb{C}^N$. By choosing Γ accordingly, and possibly shrinking U_0, we can in addition assume that each H_j is continuous and bounded on the closure of W_Γ, and in fact smooth up to $bW_\Gamma \setminus M$.

There exists another open, convex cone Γ', relatively closed in Γ, neighbourhoods $U' \subset U$ and $V' \subset V$ of $0 \in \mathbb{C}^n$ and $0 \in \mathbb{R}^d$, respectively, and $\delta = (\delta_1, \dots, \delta_d) > 0$ such that the wedge $\hat W_{\Gamma'} = \{(z,s,t) \in U' \times V' \times \Gamma' \colon 0 < t < \delta\}$ with flat edge $U' \times V'$ satisfies $\tilde W_{\Gamma'} = \Psi(\hat W_{\Gamma'}) \subset W_\Gamma$. Hence, $h_j = H_j \circ \Psi$ is well defined on $\hat W_{\Gamma'}$ for $1 \le j \le d$, extends continuously to $\hat W_{\Gamma'}$ and is smooth up to $b\hat W_{\Gamma'} \setminus U' \times V'$. Since the conclusion of the theorem is local, we can replace U by U' and V by V'. Furthermore, by shrinking the the neighbourhoods once more if necessary, we have that there exist positive constants C_1 and C_2 such that (here, $d(A,B)$ denotes the distance between a compact set A and a closed set B)

$$C_1 d((z,s,t), b\hat W_{\Gamma'}) \le d(\Psi(z,s,t), b\tilde W_{\Gamma'}) \le C_2 d((z,s,t), b\hat W_{\Gamma'}). \tag{22}$$

Our next claim is that we can replace Γ' by the standard cone $\mathbb{R}^d_+ = \{t \in \mathbb{R}^d \colon t > 0\}$. In fact, since Γ' is open, we can find d linearly independent vectors $v_1, \dots, v_j$ in Γ'. The linear mapping T which maps v_j to the j-th standard basis vector e_j is invertible, and $T^{-1}(\mathbb{R}^d_+) \subset \Gamma'$ by convexity. Then we can make a

complex linear change of coordinates by setting $(z', s', t') = (z, T^{-1}s, T^{-1}t)$. Since this coordinate change is linear and there exist positive constants C_1 and C_2 with $C_1|t| \le |t'| \le C_2|t|$, (19), (20), and (21) also hold in the new coordinates. We need just one more coordinate change.

Claim 1. *There exists a $\delta > 0$, coordinates (z, s, t) and positive constants C_1 and C_2 such that $\Psi(z, s, t) \subset \tilde{W}_{\Gamma'}$ for $(z, s) \in U \times V$, $0 < t < \delta$ and $C_1|t| \le d(b\tilde{W}_{\Gamma'}, \Psi(z, s, t)) \le C_2|t|$ for $(z, s) \in U \times V$, $0 \le t \le \delta$.*

Proof. Let e_j denote the j-th standard basis vector in $\mathbb{R}^d$, $1 \le j \le d$. If $t = (t_1, \dots, t_d) \in \mathbb{R}^d_+$, then clearly $d(t, b\mathbb{R}^d_+) = \min_{j=1}^d t_j$. For $\epsilon > 0$ consider the vectors $v_j = e_j + \epsilon \sum_{l \ne j} e_l$, $1 \le j \le d$. For ϵ small enough, these are linearly independent. We now consider the linear change of coordinates given by $z' = z$, $t' = (t'_1, \dots, t'_d) \mapsto \sum_{j=1}^d t'_j v_j$, $s' = (s'_1, \dots, s'_d) \mapsto \sum_{j=1}^d s'_j v_j$. By (22) it is enough to show that there exist positive constants C_1 and C_2 such that $C_1|t'| \le d(t, b\mathbb{R}^d_+) \le C_2|t'|$. The existence of C_2 is clear. But if $\epsilon < 1$, then $d(t, b\mathbb{R}^d_+) = \min_{j=1}^d t_j = \min_{j=1}^d (t'_j + \epsilon \sum_{l \ne j} t'_l) \ge \epsilon(t'_1 + \dots t'_d) \ge \frac{\epsilon}{d}|t|$. An appropriate choice for δ finishes the argument. □

We are going to use the notation introduced in section 2; that is, we let $\Omega_+ = U \times V \times \{t \in \mathbb{R}^d_+ : 0 < t < \delta\}$. We let $h_j = H_j \circ \Psi$ on Ω_+.

Claim 2. *$h_j \in \mathfrak{A}_\infty(\Omega_+)$ for $1 \le j \le N'$.*

Proof. By all the choices above, h_j satisfies the smoothness assumptions. Let us first check that every derivative of h_j is of slow growth. Since H_j is holomorphic in $\tilde{W}_{\Gamma'}$ and continuous on its closure, the Cauchy estimates imply that we have an estimate of the form

$$|\partial^\beta H_j(Z)| \le C_\beta (d(Z, b\tilde{W}_{\Gamma'}))^{-|\beta|} \tag{23}$$

for each β, where ∂^β denotes $\frac{\partial^{|\beta|}}{\partial Z^\beta}$. By the chain rule, $D^\alpha_{x,y,s} h_j(z, s, t)$ is a sum of products of derivatives of Ψ (which are bounded) and a derivative of H_j with respect to Z, evaluated at $\Psi(z, s, t)$, of order at most $|\alpha|$. Hence, by (23) and claim 1 we conclude that there exists a positive constant C such that

$$|D^\alpha_{x,y,s} h_j(z, s, t)| \le C_\alpha |t|^{-|\alpha|}. \tag{24}$$

We now have to estimate the derivatives of $\bar{\partial}_m h_j$ for $1 \le m \le d$. But $\bar{\partial}_m h_j = \sum_{l=1}^{N'} \frac{\partial H_j}{\partial Z_l} \bar{\partial}_m \Psi_l$. Hence, if we take an arbitrary derivative of $\bar{\partial}_m h_j$, we get a sum of products of derivatives of components of Ψ and a derivative of H_j with respect to Z each of which contains a term of the form $\bar{\partial}_m \Psi_l$. By (23) and (21) we conclude that for each compact set $K \subset U \times V$ and each $k \in \mathbb{N}$ there exists a positive constant C_k with $|D^\alpha_{x,y,s} \bar{\partial}_m h_j(z, s, t)| \le C_k |t|^k$. This proves claim 2. □

We now equip $U \times V$ with the CR-structure of M; that is, a basis of the CR-vector fields near 0 is given by $\Lambda_j = \Psi^* L_j$ for $1 \le j \le n$. We almost holomorphically extend the coefficients of the Λ_j to get smooth vector fields on an open subset of $\mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d$ containing 0.

Claim 3. *For each j, $1 \le j \le N'$, there exists a smooth function $\phi_j(Z', \bar{Z}', W)$ defined in an open neighbourhood of $(0, (\Lambda^\alpha h(0))_{|\alpha| \le k_0})$ in $\mathbb{C}^N \times \mathbb{C}^{K(k_0)}$ $(K(k_0)$*

denoting $N'|\{\alpha\colon |\alpha|\leq k_0\}|$*) such that*

$$h_j(z,s,0)=\phi_j(h(z,s,0),\overline{h(z,s,0)},(\overline{\Lambda^\alpha h(z,s,0)})_{|\alpha|\leq k_0}); \tag{25}$$

here, we write $h=(h_1,\dots,h_{N'})$*. Furthermore, after possibly shrinking* U *and* V*, the right hand side of* (25) *defines a function in* $\mathfrak{A}(\Omega_-)$*.*

This last claim of course establishes Theorem 2; since $h_j\in\mathfrak{A}(\Omega_+)$ by Claim 2 and by Claim 3 $h_j\in\mathfrak{A}(\Omega_-)$, we can apply Theorem 8 to see that h_j is smooth.

Proof. By the chain rule, we have smooth functions $\Phi_{l,\alpha}(Z',\bar Z',W)$ for $|\alpha|\leq k_0$, $1\leq l\leq d'$, defined in a neighbourhood of $\{0\}\times\mathbb{C}^{K(k_0)}$ in $\mathbb{C}^N\times\mathbb{C}^{K(k_0)}$, polynomial in the last $K(k_0)$ variables, such that

$$\Lambda^\alpha\rho_l'(h,\bar h)(z,s,0)=\Phi_{l,\alpha}(h(z,s,0),\overline{h(z,s,0)},(\Lambda^\alpha\overline{h(z,s,0)})_{|\alpha|\leq k_0}), \tag{26}$$

and $\Lambda^\alpha\rho'_{l,Z'}(h,\bar h)|_0=\Phi_{l,\alpha,Z'}(0,0,(\Lambda^\alpha h(0,0,0))_{|\alpha|\leq k_0}))$. By Definition 1 we can choose $\alpha^1,\dots,\alpha^{N'}$ and $l^1,\dots,l^{N'}$ such that if we set $\Phi=(\Phi_{l^1,\alpha^1},\dots,\Phi_{l^{N'},\alpha^{N'}})$, then $\Phi_{Z'}(0)$ is invertible. Hence, we can apply Theorem 10; let us call the solution ϕ. Then ϕ_j satisfies (25), and we shrink U and V and choose δ in such a way that $g_j(z,s,t)=\phi_j(h(z,s,-t),\overline{h(z,s,-t)},(\Lambda^\alpha\overline{h(z,s,-t)})_{|\alpha|\leq k_0})$ is well defined and continuous in a neighbourhood of $\bar\Omega_-$. It is easily checked that g_j is a function in $\mathfrak{A}(\Omega_-)$ as a consequence of (16) and the fact that each $h_j\in\mathfrak{A}_\infty(\Omega_+)$. First note that this implies $\overline{h_j(z,s,-t)}\in\mathfrak{A}_\infty(\Omega_-)$, and by Lemma 7, $\Lambda^\alpha\overline{h_j(z,s,-t)}\in\mathfrak{A}_\infty(\Omega_-)$ for each α. Now, each derivative D^β of g_j is a sum of products of derivatives of ϕ_j (which are uniformly bounded on Ω_-) and derivatives of h, $\bar h$, and $\Lambda^\alpha\bar h$, all of which fulfill the analog of (2) on Ω_-. So g_j fulfills the analog of (9) on Ω_-. Next, we compute the derivative of g_j with respect to $\bar w_k$. We have that

$$\frac{\partial g_j}{\partial\bar w_k}=\sum_{l=1}^{N\prime}\frac{\partial\phi_j}{\partial Z_l'}\frac{\partial h_l}{\partial\bar w_k}+\sum_{l=1}^{N\prime}\frac{\partial\phi_j}{\partial\bar Z_l'}\frac{\partial\bar h_l}{\partial\bar w_k}+\sum_{|\alpha|\leq k_0}\frac{\partial\phi}{\partial W_\alpha}\frac{\partial\Lambda^\alpha\bar h}{\partial\bar w_k}.$$

Applying any derivative D^β, we see that the first sum gives rise to products of derivatives of $\frac{\partial\phi_j}{\partial Z_l'}$ and derivatives of h, $\bar h$, and $\Lambda^\alpha\bar h$. Now the derivatives of ϕ_j fulfill (16). Since on $t=0$, $h=\phi(h,\bar h,(\Lambda^\alpha\bar h)_{|\alpha|\leq k_0})$, we conclude that $h-\phi(h,\bar h,(\Lambda^\alpha\bar h)_{|\alpha|\leq k_0})=O(|t|)$. But by (16), any derivative of $\frac{\partial\phi_j}{\partial\bar Z_l}(Z,\bar Z,W)$ is $O(|Z-\phi(Z,\bar Z,W)|^\infty)$, so that derivatives of $\frac{\partial\phi_j}{\partial Z_l'}$ evaluated at $(h,\bar h,(\Lambda^\alpha\bar h)_{|\alpha|\leq k_0})$ are $O(|t|^\infty)$. All the other terms in the product are $O(|t|^{-s})$ for some s, so that the terms coming from the first sum are actually $O(|t|^\infty)$. For the second and third sum, a similar argument using that $\bar h$ and $\Lambda^\alpha\bar h$ are in $\mathfrak{A}_\infty(\Omega_-)$ implies that all the terms arising from them are $O(|t|^\infty)$. All in all, we conclude that $g_j\in\mathfrak{A}_\infty(\Omega_-)$, which finishes the proof. □

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